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Existence theorems for PDEs modeling erosion and the optimal transportation of sediment

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Abstract

We prove the existence of unique global weak solutions to equations describing the sediment flow in the evolution of fluvial land surfaces, with constant water depth. These equations describe the so-called transport-limited situation, where all the sediment can be transported away given enough water. This is in distinction to the detachment-limited situation where we must wait for rock to weather (to sediment) before it can be transported away. Earlier work shows that these equations describe the optimal transport of sediment and the evolution of the surfaces in optimal transport theory. The existence theory is also extended to include diffusion in the water and the land surfaces.

Keywords Erosion · Calculus of Variations · Nonlinear Parabolic Equations · De Giorgi Minimizing Movement Scheme

Mathematics Subject Classification 49J20 · 35G20 · 35K55

1 Introduction

Water flow is a fundamental force in landscape evolution. In fluvial landscapes, erosion by water plays a foundational role in the emergence of valleys and mountains and in the formation of the fine structure of the surface. In a series of papers [1–3], Birnir, Merchant and Smith developed a mathematical model for transport-limited erosion, see [21], based on the work of Smith and Breatherton [37] and going back to Horton [5]. The first two papers are computational but in the third paper [3] the initial value

Dedicated to Professor Lars-Erik Persson on the occasion of his 80th birthday.

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problem is clarified and the scaling structure of solutions is found. The scaling turns out to be Hack's law [4], which describes the shape of river basins. All historically-known scaling properties of land surfaces and river networks [5] can be derived from Hack's law. In [3] the theory is shown to reproduce observable properties from digital elevation models for desert environments.

In [7], Birnir, Hernández, and Smith develop a mathematical model which views transport-limited erosion as a sequence of processes driven by different types of noise. They are also able to prove existence of an invariant measure for each process, thereby permitting computation of all the relevant statistical quantities. This was accomplished by linearizing the equations about the surfaces computed in [3]. The statistical phases are identified depending on the age of the surfaces, channelizing surfaces, young (adolecent) surfaces and mature surfaces. The scalings are different in each phase but in the last mature face (that is commonly called the landscape) Hack's law emerges.

Detachment limited erosion, where limiting factors on sediment transport include the resistance of the bedrock and shear stress from the water flux, was studied by Izumi and Parker [23–25] and Izumi and Fujii [22]. Binard, Degond, and Noble in [6] added a diffusion term for the soil to their model of the time evolution of the bottom topography. This term is chosen to be small in comparison with the erosion and sedimentation terms and plays an important role in the well-posedness of their model. In particular, they examine mechanisms for pattern formation by investigating the spectral stability of constant state solutions. In contrast to our approach, they begin with model equations which include a term describing the diffusion term for the soil. Their model does not factor in stochastic terms.

The model equations developed by Birnir et al. in their study of erosion are highly nonlinear. The initial value problem stated in [3] is unstable and this made the numerical integration of the equation impossible until new methods were found in [1, 2]. The theoretical work then led to the development of better numerical methods in [9]. However, the integration is still too slow and an existence theory may lead to faster numerical methods. Building on the theoretical insights, Birnir and Rowlett initiated the full nonlinear analysis of these equations by showing existence and uniqueness of entropy solutions for initial integrable data. In addition, Birnir and Rowlett identified a class of solutions which implements the optimal transportation of sediment [8]. These are the scaling solutions in the mature phase of the landscape evolution that give rise to Hack's law [7]. They also showed that if weak solutions exist, then they are unique.

To obtain solvability for these model equations, we initiate a development of existence theorems for second order nonlinear parabolic equations in divergence form for these erosion models. Solvability for second order linear elliptic equations in divergence form was studied by Trudinger in [40]. Via intermediate Schauder estimates, Gilbarg and Hörmander demonstrated existence, uniqueness, and regularity theorems for solutions of second order linear elliptic initial-boundary value problems [20], while Lieberman developed a parabolic analog of their results [28]. Examples by Dong and Kim [18], Krylov [27], Maugeri et al. [29], Meyers [30], Nadirashvili [31], Piccinini and Spagnolo [33], Talenti [39], and Ural'ceva [41] imply a general solvability theory for uniformly elliptic operators with bounded and measurable coefficients may not exist, thereby motivating consideration of particular types of discontinuous coefficients [17]. In particular, L_p solvability theorems have been demonstrated for discontinuous coefficients which belong to the class of vanishing mean oscillations (VMO). For second order equations with VMO leading coefficients, results were obtained by Chiarenza, Frasca, and Longo for nondivergence elliptic equations [15], and established by Bramante and Cerruti for linear parabolic operators with symmetric and uniformly elliptic principal part [13]. Ragusa studied divergence form elliptic equations that are also quasilinear and obtained existence, uniqueness, and regularity results when the coefficients of the principal part are in VMO and the remaining terms satisfy some linear growth conditions [35]. Ragusa likewise obtained existence, uniqueness, and regularity results for the solution of a Cauchy-Dirichlet problem associated with a nondivergence form parabolic equation with VMO coefficients [34]. In the case when the coefficients are in VMO with respect to the space variables and are merely measurable in the time variable, Krylov gave a unified approach for both divergence and nondivergence form linear elliptic and parabolic equations in the whole space in [26]. In the absence of a VMO condition on the coefficients, Bonafede has achieved existence theorems for nonlinear elliptic equations in divergence form with discontinuous coefficients [11, 12]. To develop existence theorems in the context of nonlinear parabolic equations, we extend minimizing movement schemes in the sense of de Giorgi beyond the Lebesgue measure setting (see Sect 3).

Fundamental scaling laws such as Hack's Law are known to characterize fluvial land surfaces, see [3]. The theory presented by Birnir et al. in [7] associates a spatial roughness scaling exponent to each process shaping the formation of general landscapes. In their analysis, they suggest the associated spatial roughness scaling exponents should lie in a neighborhood of 0.5 and 0.7, and that this exponent should grow to 0.75 for older landscapes. This was confirmed numerically using the improved numerical methods in [9]. In fact, our existence theory enables us to conclude that this bound on the roughness of a maturing landscape is sharp (see Corollaries 3.20 and 3.21) .

2 The Mathematical model of a transport-limited erosion process

In their mathematical study of transport-limited erosion, Birnir and Rowlett [8] model a mountain ridge defined over a rectangular domain of length L and width W, see [3], by setting

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le W, 0, \le y \le L\}.$$

As in their work, let $H = z + h$ denote the height of the water surface and h the water depth. z is the height of the land surface.The conditions on the lateral boundaries of the ridge at $y = 0$ and $y = L$ are taken to be periodic so that the linear ridge modeled is of infinite extent. The top of the ridge, located at $x = 0$, corresponds to a water depth of zero and the base of the ridge, located at $x = W$, to an absorbing body of water. These assumptions are encoded in the following boundary conditions:

$$|\nabla H(0, y, t)|^2 \, h(0, y)^{10/3} \, \nabla H(0, y, t) \cdot n = 0,$$
$$h(x, 0) = h(x, L),$$

$$H(x, 0, t) = H(x, L, t),$$
$$H(W, y, t) = 0. \tag{1}$$

The first one of these conditions says that there is no sediment flow over the top of the ridge and the last one that the sediment is absorbed by a river or a lake at the lower boundary, at $x = W$. The initial conditions for the sediment are taken to be a flat ridge sloping in the x direction with some slope c (grade) of the ridge,

$$H_0(x, y) = H(x, y, 0) = c(W - x), \quad 0 \le x \le W, 0, \le y \le L.$$

Based on numerical evidence, see Fig 2, and the discussion below, a statistically stationary water depth is also assumed to exist. The water flow and the sediment flow in their respective models, see [1, 8], each have different time scales. The sediment flow is associated with a larger time scale and is described by

$$\frac{\partial H}{\partial t} = \nabla \cdot [\nabla H |\nabla H|^2 h^{10/3}]. \tag{2}$$

The water depth h is viewed as a non-negative quantity which does not depend on time. More precisely,

$$h \ge 0, \ h > 0 \text{ a.e. on } S, \ h^{-10/9} \in L^1(S), \ h \in L^\infty(\Omega), \tag{3}$$

where $S \subset \Omega$ is a piecewise smooth domain contained in Ω.

The reasons for the choice of the water depth h are the following. The timescales for a significant amount of water to flow and a similar amount of sediment to flow are very different. The water time scale is much faster and this requires that in the numerical integrations many time steps are integrated for the water equations, see [3] and [9], for each integration of the sediment equations. In fact, one allows the water surface to settle down to an equilibrium surface $h(x, y)$ before taking one step in the sediment flow. The existence of the solutions of the water equations in [3] follows from the theory of hyperbolic conservation laws. The reason is that the water flows down the gradient of the sediment surface and the equations become one-dimensional. They are integrated and analyzed in Welsh, Birnir and Bertozzi [10]. The solutions are not smooth but contain shocks, carving out the mountain slopes, as well as hydrolic jumps at the base. Thus one can only assume that the equilibrium water surface h is a measurable function in the appropriate function space.

Birnir and Rowlett show that weak solutions, when they exist, are unique in L^2. In this paper, we continue development of the mathematical analysis of these equations by proving existence of weak solutions. Our main result is given in the following theorem.

Theorem 2.1 *Let $h(x, y)$ be a given function which satisfies the conditions described in (3). Then, for any $H_0 \in W_h^{1,4}(\Omega)$, there exists a unique weak solution, see Sect. 3.1, $H \in L^2([0, \infty); W_h^{1,4}(\Omega))$ to the model eq (2) for the sediment on $[0, \infty) \times \Omega$ and satisfying the initial and boundary conditions (1).*

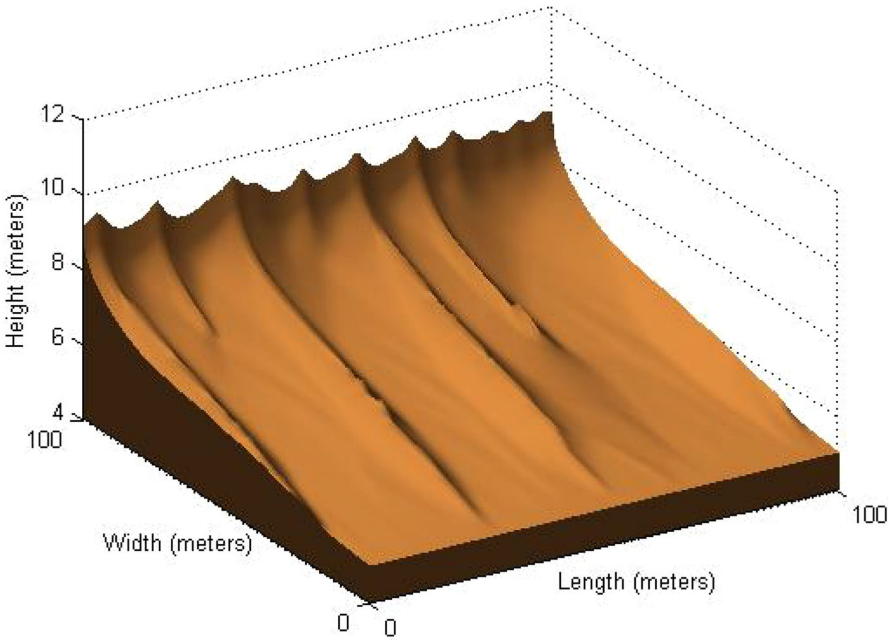

Fig. 1 A typical surface Water Surface at 60% eroded, from [9]

3 Existence of a gradient flow on a weighted sobolev space

Birnir and Rowlett show the existence of entropy solutions for eq (2) by developing their mathematical model in the setting of the weighted Sobolev space,

$$W_h^{1,4}(\Omega) := \left\{ u \in L^4(\Omega) \text{ such that } h^{5/6}\frac{\partial u}{\partial x} \in L^4(S) \text{ and } h^{5/6}\frac{\partial u}{\partial y} \in L^4(S) \right\}$$

equipped with the weighted Sobolev norm

$$\|u\|_{W_h^{1,4}(\Omega)} := \left(\int_\Omega (|u|^4 + |\nabla u|^4 h^{10/3})\, dx \right)^{1/4},$$

and the functional

$$K(u) := \int_\Omega \frac{|\nabla u|^4}{4} h^{10/3}\, dx. \tag{4}$$

Throughout this paper, dx denotes integration with respect to the standard Lebesgue measure on $\mathbb{R}^2$. Note that $h = 1$ corresponds to the usual Sobolev space $W^{1,4}(\Omega)$ and that functional (4) gives rise to the gradient flow of eq (2) with respect to the L^2 Euclidean structure. For our work on transport-limited erosion and gradient flows, we simplify the presentation of our estimates by working instead with the weighted

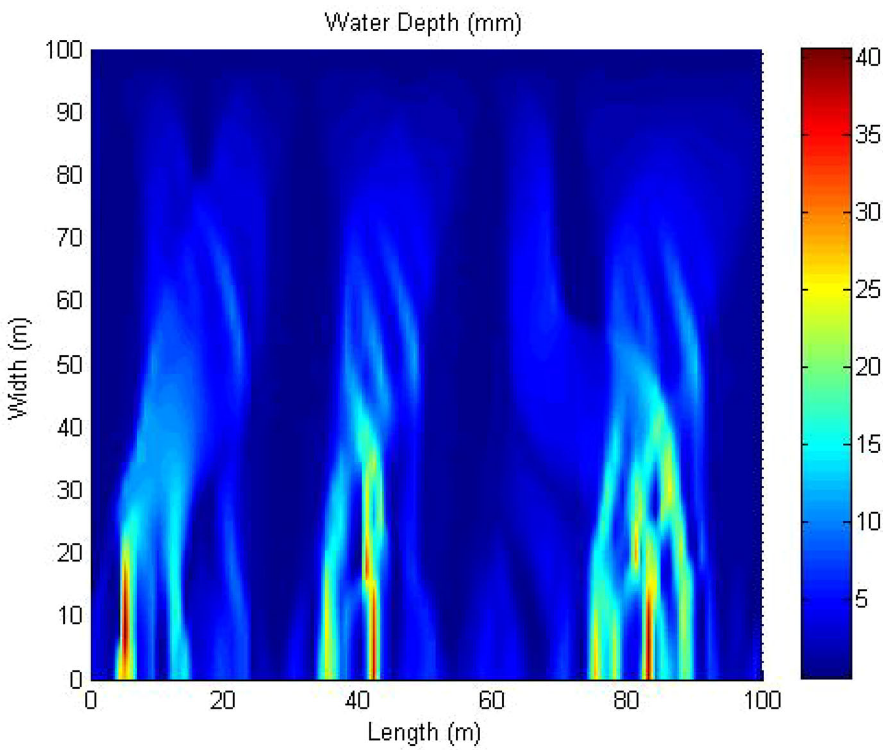

Fig. 2 The depth of water h over the surface in Fig. 1, from [9]

Banach space,

$$\widetilde{W}_h^{1,4}(\Omega) := \left\{ u \in L^4_{h^{10/3}}(\Omega) \text{ such that } h^{5/6}\frac{\partial u}{\partial x} \in L^4(S) \text{ and } h^{5/6}\frac{\partial u}{\partial y} \in L^4(S) \right\}$$

equipped with the norm

$$\|u\|_{\widetilde{W}_h^{1,4}(\Omega)} := \left(\int_\Omega (|u|^4 + |\nabla u|^4)\, h^{10/3} dx \right)^{1/4}.$$

We shall also work with the weighted L^2 space,

$$L^2_{1/\tau}(\Omega) := \left\{ u \in L^2(\Omega) \text{ such that } \frac{1}{\tau} u \in L^2(S) \right\},$$

where τ is a fixed positive real number. We will demonstrate by construction the existence of weak solutions to eq (2). More precisely, our solutions satisfy the following conditions.

Definition 3.1 ([8]) Given h which satisfies (3), assume further that h is continuous on $\Omega \setminus h^{-1}(0)$ and that $h^{-1}(0)$ is a finite union of piecewise smooth curves. Then a **weak solution** of (2) on $(0, T)$ with initial condition $H_0 \in \widetilde{W}_h^{1,4}(\Omega)$ is

$$H \in \widetilde{W}_h^{1,4}((0,T);\Omega)$$

which satisfies

$$\begin{aligned} &H = H_0 \text{ a. e. on } \Omega, \qquad t = 0;\\ &\int_\Omega \Big(\frac{\partial H}{\partial t}\phi + h^{10/3}|\nabla H(t)|^2 \nabla H(t)\cdot\nabla\phi\Big)\,dx = 0, \qquad t \in (0,T); \end{aligned}$$

for all $\phi \in C_0^\infty(\Omega \setminus h^{-1}(0))$.

Remark 3.2 Since the mountain ridges are modeled by a finite union of piecewise smooth curves, $h^{-1}(0)$ corresponds to the top of the ridges. The sediment flow at the top of the ridge is assumed to be discontinuous because sediment flows down either side of the ridge. Based on such physical considerations, it makes sense to solve the equation away from $h^{-1}(0)$ and use test functions with compact support in $\Omega \setminus h^{-1}(0)$.

Subspaces of $\widetilde{W}_h^{1,4}(\Omega)$ where the boundary conditions (1) are satisfied support a Poincaré inequality. This property will be useful for upper bound estimates.

Lemma 3.3 *Let h be a given function which satisfies the conditions described in (3). If $u \in \widetilde{W}_h^{1,4}(\Omega)$ satisfies the boundary conditions (1), then there exists $C_h > 0$ such that*

$$||u||_{L^4_{h^{10/3}}(\Omega)} \le C_h ||\nabla u||_{L^4_{h^{10/3}}(\Omega)}. \tag{5}$$

Proof Since u vanishes when $x = W$,

$$\begin{aligned} |u(x,y,t)| = |u(x,y,t) - u(W,y,t)| &\le \Big|\int_x^W \frac{\partial u}{\partial x_1}\,dx_1\Big|\\ &\le \int_x^W \Big|\frac{\partial u}{\partial x_1}\Big|\,dx_1\\ &\le \int_0^L\int_0^W h^{-5/6}h^{5/6}\Big|\frac{\partial u}{\partial x_1}\Big|\,dx\\ &\le \Big(\int_\Omega (h^{-5/6})^{4/3}\Big)^{3/4}\Big(\int_\Omega \Big|h^{5/6}\frac{\partial u}{\partial x_1}\Big|^4\,dx\Big)^{1/4}\\ &\le C_h'\Big(\int_\Omega |\nabla u|^4 h^{10/3}\,dx\Big)^{1/4} \end{aligned}$$

for some constant $C_h' > 0$, hence

$$\int_\Omega |u|^4 h^{10/3}\,dx \le ||h||_\infty^{10/3}\int_\Omega (C_h')^4 \int_\Omega |\nabla u|^4 h^{10/3}\,dx$$

$$\leq ||h||_\infty^{10/3}(C_h')^4|\Omega| \int_\Omega |\nabla u|^4 h^{10/3}\, dx.$$

Therefore, inequality (5) holds for $C_h \geq 1 + ||h||_\infty^{5/6} C_h' |\Omega|^{1/4}$. □

Remark 3.4 Lemma 3.3 will allow us to consider only the gradient norm when working with functions in $\widetilde{W}_h^{1,4}(\Omega)$ which satisfy the boundary conditions (1).

Our construction of a weak solution to eq (2) is based on a Minimizing Movement Scheme in the sense of De Giorgi (see [36] for an exposition) and will rely on the existence of minimizers for the functional K. Techniques from direct methods in the calculus of variations will be used to obtain existence of a sequence of minimizers which each belong to $\widetilde{W}_h^{1,4}(\Omega)$.

In Proposition 3.15, we perturb the integrand of the functional K by a function in $L^2_{1/\tau}(\Omega)$ and obtain the existence of a minimizer which satisfies our boundary conditions (1). Our proof of this result requires that our function spaces possess some compactness properties.

Proposition 3.5 *(Kakutani, Theorem 3.17 from [14]) Let E be a Banach space. Then E is reflexive if and only if $B_E := \{x \in E : ||x||_E \leq 1\}$ is weakly compact.*

Since B_E is convex, this set is also weakly closed. In the Banach space setting, the Eberlein-Šmulian Theorem gives that a weakly closed set is weakly compact if and only if the set is weakly sequentially compact [16]. Consequently, a Banach space is reflexive if and only if its unit ball is weakly sequentially compact. In particular,

Proposition 3.6 *(Theorems 3.18 and 3.19 from [14]) Assume that E is a Banach space. Then E is reflexive if and only if every bounded sequence in E admits a weakly convergent subsequence.*

Lemma 3.7 *Let h be a given function which satisfies the conditions described in (3). Then $\widetilde{W}_h^{1,4}(\Omega)$ is a reflexive Banach space.*

Proof Recall that $W^{1,4}(\Omega)$ is a reflexive Sobolev space. By Proposition 3.6, every bounded sequence in $W^{1,4}(\Omega)$ admits a weakly convergent subsequence. To show that $\|\cdot\|_{W^{1,4}(\Omega)}$ is equivalent to $\|\cdot\|_{\widetilde{W}_h^{1,4}(\Omega)}$ on $\widetilde{W}_h^{1,4}(\Omega)$, note that if $u \in \widetilde{W}_h^{1,4}(\Omega)$, then

$$\|u\|^4_{W^{1,4}(\Omega)} = \int_\Omega (|u|^4 + |\nabla u|^4) h^{10/3} h^{-10/3}\, dx \overset{(3)}{\leq} ||h||_\infty^{-10/3} \int_\Omega (|u|^4 + |\nabla u|^4) h^{10/3}\, dx$$
$$\leq ||h||_\infty^{-10/3} ||u||^4_{\widetilde{W}_h^{1,4}(\Omega)} < \infty.$$

Application of (3) also gives

$$\int_\Omega (|u|^4 + |\nabla u|^4) h^{10/3}\, dx \leq ||h||_\infty^{10/3} \int_\Omega (|u|^4 + |\nabla u|^4)\, dx.$$

Together, both sets of bounds imply

$$\frac{1}{1+||h||_{\infty}^{10/3}}\|u\|^4_{\widetilde{W}_h^{1,4}(\Omega)} \le \|u\|^4_{W^{1,4}(\Omega)} \le (1+||h||_{\infty}^{-10/3})||u||^4_{\widetilde{W}_h^{1,4}(\Omega)}.$$

As a consequence, every sequence which is bounded in $\widetilde{W}_h^{1,4}(\Omega)$ is also bounded in $W^{1,4}(\Omega)$ and so admits a weakly convergent subsequence. In particular, $\widetilde{W}_h^{1,4}(\Omega)$ is a reflexive Banach space. □

The proof of Lemma 3.7 also gives a basis for the following level of equivalence between $W_h^{1,4}(\Omega)$ and $\widetilde{W}_h^{1,4}(\Omega)$.

Lemma 3.8 *Let h be a given function which satisfies the conditions described in (3). Then the norms $\|\cdot\|_{\widetilde{W}_h^{1,4}(\Omega)}$ and $\|\cdot\|_{W_h^{1,4}(\Omega)}$ are equivalent on functions in $W_h^{1,4}(\Omega)$ which satisfy boundary conditions (1).*

Proof By the proof of Lemma 3.7, there exist constants $C_1,\ C_2$, such that for every $u \in W_h^{1,4}(\Omega)$,

$$C_1\|u\|_{\widetilde{W}_h^{1,4}(\Omega)} \le \|u\|_{W_h^{1,4}(\Omega)} \le C_2\|u\|_{\widetilde{W}_h^{1,4}(\Omega)}.$$

□

Many of our arguments will take place in the setting of the following Banach space.

Lemma 3.9 *Let h be a given function which satisfies the conditions described in (3). Then for fixed $\tau > 0$, $\widetilde{W}_h^{1,4}(\Omega) \cap L^2_{1/\tau}(\Omega)$ is a reflexive Banach space.*

Proof By Lemma 3.7, $\widetilde{W}_h^{1,4}(\Omega)$ is a reflexive Banach space. Since every closed linear subspace of a reflexive Banach space is also a reflexive Banach space ([14], Proposition 3.20), $\widetilde{W}_h^{1,4}(\Omega) \cap L^2_{1/\tau}(\Omega)$ is a reflexive Banach space. □

3.1 Minimizing sequences in $\widetilde{W}_h^{1,4}(\Omega)$

Minimizing sequences, convex functions, and reflexive Banach spaces like $\widetilde{W}_h^{1,4}(\Omega)$ play an important role in direct methods in the calculus of variations. To define a minimizing sequence in $M \subset \widetilde{W}_h^{1,4}(\Omega)$ for K, let

$$\alpha = \inf_{u\in M} K(u).$$

Then $(u_i) \subset M$ is called a *minimizing sequence* in M for K if

$$K(u_i) \to \alpha,\ \text{as } i \to \infty.$$

Now let

$$V := \{u \in \widetilde{W}_h^{1,4}(\Omega) : u(x,0,t) = u(x,L,t), u(W,y,t) = 0,$$
$$|\nabla u(0,y,t)|^2 h(0,y)^{10/3} \nabla u(0,y,t) \cdot n = 0\},$$
$$V_\tau := \{u \in \widetilde{W}_h^{1,4}(\Omega) \cap L^2_{1/\tau}(\Omega) : u(x,0,t) = u(x,L,t), u(W,y,t) = 0,$$
$$|\nabla u(0,y,t)|^2 h(0,y)^{10/3} \nabla u(0,y,t) \cdot n = 0\},$$

and K be a functional on $\widetilde{W}_h^{1,4}(\Omega)$.

Under suitable conditions, properties of convex functions can be leveraged to obtain functionals on $\widetilde{W}_h^{1,4}(\Omega) \cap L^2_{1/\tau}(\Omega)$ which exhibit the following property.

Definition 3.10 ([19, 38]) Let X be a topological space. Then $J : X \to \mathbb{R} \cup \{\infty\}$ is called **sequentially weakly lower semicontinuous** if J is weakly lower semicontinuous and for any $x \in X$, any sequence (x_i) in X such that $x_i \rightharpoonup x$ weakly in X,

$$J(x) \le \liminf_{i\to\infty} J(x_i).$$

Note that a sequentially lower semicontinuous function achieves its infimum on X if X is a sequentially compact topological space. If K is nonnegative on $\widetilde{W}_h^{1,4}(\Omega)$, the infimum on $\widetilde{W}_h^{1,4}(\Omega)$, as well as on any $M \subset \widetilde{W}_h^{1,4}(\Omega)$, is always finite. In particular, K achieves its infimum on M if K is weakly lower semicontinous and M is weakly compact. When K is also sequentially weakly lower semicontinuous, then a weakly convergent subsequence can be extracted from any minimizing sequence in M for K.

A function in $\widetilde{W}_h^{1,4}((0,T);\Omega)$ will be built from a sequence of functions in $\widetilde{W}_h^{1,4}(\Omega)$ (see Sect 3.2). Each function in this sequence arises as the minimizer of a functional on V_τ which varies from K by a parameter (see Proposition 3.15). The existence of each such minimizer relies on the fact (which is demonstrated below) that $B_{\widetilde{W}_h^{1,4}(\Omega)\cap L^2_{1/\tau}(\Omega)} \subset \widetilde{W}_h^{1,4}(\Omega)$ is weakly compact for each $\tau > 0$.

Lemma 3.11 *Let h be a given function which satisfies the conditions described in (3) and $\tau > 0$. Then V_τ is closed and convex. In particular, every bounded sequence in V_τ admits a weakly convergent subsequence.*

Proof In infinite dimensional Banach spaces, closed and convex sets are also weakly closed. To see that V_τ is convex, suppose that u_1 and u_2 are in V_τ. Choose α in $(0,1)$. Then

$$\alpha u_1 + (1-\alpha)u_2 \in \widetilde{W}_h^{1,4}(\Omega) \cap L^2_{1/\tau}(\Omega),$$
$$\alpha u_1(x,0,t) + (1-\alpha)u_2(x,0,t) = \alpha u_1(x,L,t) + (1-\alpha)u_2(x,L,t),$$
$$\alpha u_1(W,y,t) + (1-\alpha)u_2(W,y,t) = 0.$$

Observe that $|\nabla u_1(0,y,t)|^2 h(0,y)^{10/3}$ is a scalar. In particular, the boundary conditions imply that $|\nabla u_1(0,y,t)|^2 h(0,y)^{10/3} = 0$ or $\nabla u_1(0,y,t) \cdot n = 0$. Since similar reasoning can be applied to u_2,

$$|\nabla(\alpha u_1(0,y,t) + (1-\alpha)u_2(0,y,t))|^2 h(0,y)^{10/3} \nabla(\alpha u_1(0,y,t) + (1-\alpha)u_2(0,y,t)) \cdot n$$

$$
\begin{aligned}
&= (|\nabla(\alpha u_1(0,y,t) + (1-\alpha)u_2(0,y,t))|^2\, h(0,y)^{10/3}\alpha)(\nabla u_1(0,y,t)\cdot n) \\
&\quad + (|\nabla(\alpha u_1(0,y,t) + (1-\alpha)u_2(0,y,t))|^2 h(0,y)^{10/3}(1-\alpha))(\nabla u_2(0,y,t)\cdot n) \\
&= 0.
\end{aligned}
$$

Therefore, V_τ is weakly closed. Note that every sequence which is bounded in V_τ is also bounded in $\widetilde{W}_h^{1,4}(\Omega) \cap L^2_{1/\tau}(\Omega)$. Since $\widetilde{W}_h^{1,4}(\Omega) \cap L^2_{1/\tau}(\Omega)$ is reflexive (Lemma 3.9), Proposition 3.6 gives that a weakly convergent subsequence can be extracted from every bounded sequence in V_τ. Because V_τ is weakly closed, the weak limit of such a sequence must be also in V_τ. □

Note that $V_\tau \subset V$ for every $\tau > 0$. Sequences in V which are bounded in gradient norm are also bounded in $\widetilde{W}_h^{1,4}(\Omega)$ by Lemma 3.3. In fact, the following equivalence holds:

Lemma 3.12 *Let h be a given function which satisfies the conditions described in (3) and $\tau > 0$. For $u \in \widetilde{W}_h^{1,4}(\Omega)$,*

$$
\|u\|_{\dot{\widetilde{W}}_h^{1,4}(\Omega)} := \|\nabla u\|_{L^4_{10/3}}(\Omega).
$$

Then $\|\cdot\|_{\dot{\widetilde{W}}_h^{1,4}(\Omega)}$ is equivalent to $\|\cdot\|_{\widetilde{W}_h^{1,4}(\Omega)}$ on V_τ.

Proof For every $u \in V_\tau \subset V$, Lemma 3.3 yields $C_h > 0$ such that

$$
\|u\|_{\dot{\widetilde{W}}_h^{1,4}(\Omega)} \le \|u\|_{\widetilde{W}_h^{1,4}(\Omega)} \le (1 + C_h)\|u\|_{\dot{\widetilde{W}}_h^{1,4}(\Omega)}.
$$

□

To demonstrate that K has a minimizer in V, we also work with the density $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ given by

$$
F(x, p, q) = \frac{|q|^4}{4}[h(x)]^{10/3}.
$$

Then

$$
K(u) = \int_\Omega \frac{|\nabla u(x)|^4}{4} h(x)^{10/3}\, dx = \int_\Omega F(x, u(x), \nabla u(x))\, dx.
$$

This reframing of our functional is useful for the proof of the following lemma, as well as for other results in Sect 3.2.

Lemma 3.13 *Let h be a given function which satisfies the conditions described in (3). Then K is sequentially weakly lower semicontinuous.*

Proof Take any $\lambda \in \mathbb{R}$. Then

$$M_\lambda = \{v \in V : K(v) \le \lambda\}$$

is convex and strongly closed, hence weakly closed. In particular, K is weakly lower semicontinuous. To see that K is also sequentially weakly lower semicontinuous, observe that $(p, q) \to F(x, p, q)$ can be viewed as convex for all $p \in \mathbb{R}$ and almost every $x \in \Omega$. Therefore for every $v_i \rightharpoonup v$ in V,

$$F(x, v_i, \nabla v_i) \ge F(x, v, \nabla v) + F_p(x, v, \nabla v)(v_i - v) + \langle F_q(x, v, \nabla v), \nabla v_i - \nabla v\rangle.$$

To obtain a bound on $\langle F_q(x, v, \nabla v), \nabla v_i - \nabla v\rangle$, we use Hölder's inequality,

$$\begin{aligned}\int_\Omega \langle F_q(x, v, \nabla v), \nabla v_i - \nabla v\rangle dx &\le (\int_\Omega |\nabla v|^4 h^{10/3} dx)^{3/4} (\int_\Omega |\nabla v_i - \nabla v|^4 h^{10/3} dx)^{1/4} \\ &= ||v||^{3/4}_{\overset{\bullet}{\widetilde{W}}{}^{1,4}_h(\Omega)} ||v_i - v||^{1/4}_{\overset{\bullet}{\widetilde{W}}{}^{1,4}_h(\Omega)}.\end{aligned}$$

Because $F_p(x, v, \nabla v) = 0$ and $\nabla v_i \rightharpoonup \nabla v$ in $L^4(\Omega)$, $K(v_i) \ge K(v)$. Thus

$$K(v) \le \lim\inf_{i\to\infty} K(v_i),$$

as desired. □

In Sect 3.2, a minimizer for K in V will be built from a sequence of functions in V which each minimize

$$u \longmapsto \int_\Omega \Big[\frac{|\nabla u(x)|^4}{4} h(x)^{10/3} + \frac{|u(x) - u_0(x)|^2}{2\tau}\Big] dx$$

in V_τ for some fixed $\tau > 0$ and $u_0 \in V_\tau$. Let $F^\tau : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be given by

$$F^\tau(x, p, q) = F(x, p, q) + \frac{|p|^2}{2\tau}.$$

For the next lemma, we show that

$$K^\tau(u) := \int_\Omega F^\tau(x, u(x), \nabla u(x))\, dx,$$

is sequentially lower semicontinuous.

Lemma 3.14 *Let h be a given function which satisfies the conditions described in (3) and $\tau > 0$. Then K^τ is sequentially weakly lower semicontinuous.*

Proof Take any $\lambda \in \mathbb{R}$. Then

$$M_\lambda = \{v \in V_\tau : K^\tau(v) \leq \lambda\}$$

is convex and strongly closed, hence weakly closed. In particular, K^τ is weakly lower semicontinuous. To see that K^τ is also sequentially weakly lower semicontinuous, observe that $(p, q) \to F^\tau(x, p, q)$ can be viewed as convex for all $p \in \mathbb{R}$ and almost every $x \in \Omega$. Therefore for every $v_i \rightharpoonup v$ in V_τ,

$$F^\tau(x, v_i, \nabla v_i) \geq F^\tau(x, v, \nabla v) + F_p^\tau(x, v, \nabla v)(v_i - v) + \langle F_q^\tau(x, v, \nabla v), \nabla v_i - \nabla v\rangle.$$

We get the same bound on $\langle F_q^\tau(x, v, \nabla v), \nabla v_i - \nabla v\rangle$ as above, namely,

$$\int_\Omega \langle F_q^\tau(x, v, \nabla v), \nabla v_i - \nabla v\rangle dx \leq ||v||^{3/4}_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)} ||v_i - v||^{1/4}_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)},$$

by Hölder's inequality. The inequality

$$0 \leq F^\tau(x, p, q) \leq 1 + \frac{1}{2\tau}|p|^2 + \frac{1}{4}||h||_\infty^{10/3}|q|^4$$

implies that there exist $C_3, C_4 > 0$ such that

$$\int_\Omega |F_p^\tau(x, v, \nabla v)|^2\, dx \leq C_3 \int_\Omega (1 + |v|)^2\, dx \leq C_4(1 + ||v||_{L^2_{1/\tau}(\Omega)}) < \infty.$$

Because $v_i \rightharpoonup v$ and $\nabla v_i \rightharpoonup \nabla v$ in $L^2(\Omega) \cap L^4(\Omega)$, $K^\tau(v_i) \geq K^\tau(v)$. Thus

$$K^\tau(v) \leq \liminf_{i\to\infty} K^\tau(v_i),$$

as desired. □

Proposition 3.15 *Let h be a given function which satisfies the conditions described in (3). Then for fixed $\tau > 0$ and $u_0 \in V$, the minimization problem*

$$u_t \in \; argmin_{w \in V_\tau} \int_\Omega \left[F(x, w(x), \nabla w(x)) + \frac{|w(x) - u_0(x)|^2}{2\tau}\right] dx \tag{6}$$

admits a solution in $\widetilde{W}_h^{1,4}(\Omega)$ which satisfies the boundary conditions given in (1).

Proof This minimization problem (6) is equivalent to the existence of a minimizer in V_τ for

$$K_0^\tau(w) := \int_\Omega F^\tau(x, w(x) - u_0(x), \nabla w(x))\, dx.$$

Note that arguments similar to those as in the proof of Lemma 3.14 can also be used to show that this functional is sequentially weakly lower semicontinuous if $w - u_0 \in V_\tau$ for any $w \in V_\tau$. Since $u_0 \in V \subset \widetilde{W}_h^{1,4}(\Omega)$, $h \in L^\infty(\Omega)$, and $\tau > 0$ is fixed,

$$\begin{aligned}\int_\Omega \frac{1}{\tau}|u_0|^2\,dx &\le \frac{1}{\tau}\Big(\int_\Omega |u_0|^4 h^{10/3}\,dx\Big)^{1/2}\Big(\int_\Omega h^{-10/3}\,dx\Big)^{1/2}\\ &\le \frac{1}{\tau}||u_0||^2_{\widetilde{W}_h^{1,4}(\Omega)}(||h||_\infty^{-10/3})^{1/2} < \infty.\end{aligned}$$

In particular, $u_0 \in L^2_{1/\tau}(\Omega)$, as is $w - u_0$ for any $w \in V_\tau$. Now take any minimizing sequence (w_i) for K_0^τ, in a closed ball $\overline{B}$ in V_τ, with initial value u_0. This sequence admits a subsequence which weakly converges to some $w \in V_\tau$. Because K_0^τ is nonnegative and sequentially weakly lower semicontinuous,

$$0 \le K_0^\tau(w) \le \lim\inf_{i\to\infty} K_0^\tau(w_i).$$

As every function in V_τ is in $\widetilde{W}_h^{1,4}(\Omega)$ and satisfies the boundary conditions given in (1), the desired minimizer for K_0^τ exists. □

3.2 A De Giorgi minimizing movement scheme

In [36], Santambrogio describes how a Cauchy problem can be extended to the metric (and hence the Wasserstein) setting. The solution of the Cauchy problem involves De Giorgi Minimizing Movement Schemes. To build a function in $\widetilde{W}_h^{1,4}((0,T);\Omega)$ which is a weak solution to (2) and satisfies the Cauchy problem (2) and the boundary conditions (1), we extend De Giorgi Minimizing Movement Schemes to the weighted Sobolev space setting. We begin by obtaining a sequence of functions in $\widetilde{W}_h^{1,4}(\Omega)$ and constructing an interpolation between these functions. We facilitate the study of our functionals in this gradient flow framework by now viewing our density F in the formulation of $K : \widetilde{W}_h^{1,4}(\Omega) \to \mathbb{R}$ as

$$K(u) = \int_\Omega \frac{|\nabla u(x,t)|^4}{4} h(x)^{10/3}\,dx := \int_\Omega F(\nabla u(x,t))\,dx.$$

Proposition 3.16 *Choose $T > 0$ and partition $[0,T]$ into time steps of size $\tau > 0$. For $k = 0$, take $u_0^\tau(x,t_0) \in V$. For each positive integer k, there exists a solution to*

$$u_{k+1}^\tau \in argmin_{w\in V_\tau} \int_\Omega \Big[F(\nabla w(x,(k+1)\tau)) + \frac{|w(x,(k+1)\tau) - u_k^\tau(x,k\tau)|^2}{2\tau}\Big]dx. \quad (7)$$

Furthermore,

$$\infty > ||u_0^\tau(x,t_0)||^4_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)} = \int_\Omega F(\nabla u_0^\tau(x,t_0))\,dx \geq \int_\Omega F(\nabla u_1^\tau(x,\tau))\,dx \geq$$

$$\cdots \geq \int_\Omega F(\nabla u_k^\tau(x,k\tau))\,dx$$

$$\geq \int_\Omega F(\nabla u_{k+1}^\tau(x,(k+1)\tau))\,dx \geq \cdots 0, \tag{8}$$

and each u_{k+1}^τ at some fixed $t \in [0,T]$ satisfies

$$\frac{u_{k+1}^\tau - u_k^\tau}{\tau} = \frac{\delta F(\nabla w)}{\delta w}[u_{k+1}^\tau] = \nabla\cdot[\nabla u_{k+1}^\tau\,|\nabla u_{k+1}^\tau|^2] \quad \textit{for} \quad \textit{a.e.} \quad \textit{x}. \tag{9}$$

Proof For $k = 0$, Proposition 3.15 gives the existence of a solution u_t to minimization problem (7) after setting $u_0^\tau(x,t_0) = u_0(x)$, and $w(x,\tau) = w(x)$. In particular, any minimizing sequence $(\, w_i(x,\tau)\,)$ in V_τ for K_0^τ with initial value $w_0(x,\tau) = u_0^\tau(x,t_0)$ weakly converges to $w(x,\tau)$ for some $w \in V_\tau$. Letting $u_1^\tau(x,\tau) := w(x,\tau)$ yields

$$\infty > ||u_0^\tau(x,t_0)||^4_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)} = K(u_0^\tau(x,\tau)) = \int_\Omega F(\nabla u_0^\tau(x,t_0))\,dx$$

$$= \int_\Omega \Big[F(\nabla u_0^\tau(x,t_0)) + \frac{|u_0^\tau(x,t_0) - u_0^\tau(x,t_0)|^2}{2\tau}\Big]\,dx$$

$$= K_0^\tau(u_0^\tau(x,t_0)) \geq K_0^\tau(u_1^\tau(x,\tau))$$

$$= \int_\Omega \Big[F(\nabla u_1^\tau(x,\tau)) + \frac{|u_1^\tau(x,\tau) - u_0^\tau(x,t_0)|^2}{2\tau}\Big]\,dx$$

$$\geq \int_\Omega F(\nabla u_1^\tau(x,\tau))\,dx = K(u_1^\tau(x,\tau)) \geq 0.$$

Moreover, at some fixed $t \in [0,T]$, u_1^τ satisfies

$$\frac{u_1^\tau - u_0^\tau}{\tau} = \frac{\delta F(\nabla w)}{\delta w}[u_1^\tau] = \nabla\cdot[\nabla u_1^\tau\,|\nabla u_1^\tau|^2] \text{ for a.e. x.}$$

Now let m be some fixed positive integer and suppose that for all positive integers $k \leq m$, there exists a solution to minimization problem (7), the inequalities in (8) hold, and that each such solution satisfies (9). For the case of $k = m+1$, minimization problem (7) is equivalent to the existence of a minimizer in V_τ for

$$K_m^\tau(w(x,(m+1)\tau)) := \int_\Omega F^\tau(x, w(x,(m+1)\tau) - u_m^\tau(x,m\tau), \nabla w(x,(m+1)\tau))\,dx.$$

Note that $u_m^\tau \in V_\tau$, hence $w(x,(m+1)\tau) - u_m^\tau(x, m\tau) \in L^2_{1/\tau}(\Omega)$ for any $w \in V_\tau$. Therefore, arguments similar to those as in the proof of Lemma 3.14 can also be used to show that this functional is sequentially weakly lower semicontinuous. Now take any minimizing sequence $(w_i(x,(m+1)\tau))$ in V_τ for K_m^τ with initial value $w_0(x,(m+1)\tau) = u_m^\tau(x, m\tau)$. Up to extraction of a subsequence,

$$\begin{aligned}
0 &\le ||w_i(x,(m+1)\tau)||_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)} = (K(w_i(x,(m+1)\tau)))^{1/4} \\
&\le (K_m^\tau(w_i(x,(m+1)\tau)))^{1/4} \\
&\le (K_m^\tau(u_m^\tau(x,m\tau)))^{1/4} = (K(u_m^\tau(x,m\tau)))^{1/4} \\
&\le (K_{m-1}^\tau(u_m^\tau(x,m\tau)))^{1/4} \le (K_{m-1}^\tau(u_{m-1}^\tau(x,(m-1)\tau)))^{1/4} \\
&= (K(u_{m-1}^\tau(x,(m-1)\tau)))^{1/4} \le \cdots \\
&\cdots \le (K(u_1^\tau(x,\tau)))^{1/4} \\
&\le (K(u_0^\tau(x,t_0)))^{1/4} = ||u_0^\tau(x,t_0)||_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)}
\end{aligned}$$

for all nonnegative i. Lemma 3.12 gives that this subsequence is also bounded in V_τ. By Lemma 3.12, this subsequence admits another subsequence which weakly converges to $u_{m+1}^\tau(x,(m+1)\tau) := w(x,(m+1)\tau)$ for some $w \in V_\tau$. Because K_m^τ is nonnegative and sequentially weakly lower semicontinuous,

$$\begin{aligned}
0 &\le K_m^\tau(u_{m+1}^\tau(x,(m+1)\tau) \le \liminf_{i\to\infty} K_m^\tau(w_i(x,(m+1)\tau) \\
&\le K_m^\tau(u_m(x,m\tau)) = K(u_m(x,m\tau)),
\end{aligned}$$

the desired minimizer for K_m^τ exists. Furthermore,

$$\begin{aligned}
\infty > &||u_0^\tau(x,t_0)||^4_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)} = \int_\Omega F(\nabla u_0^\tau(x,t_0))\,dx \ge \int_\Omega F(\nabla u_1^\tau(x,\tau))\,dx \ge \\
&\cdots \ge \int_\Omega F(\nabla u_{m-1}^\tau(x,(m-1)\tau))\,dx \\
&\ge \int_\Omega F(\nabla u_m^\tau(x,m\tau))\,dx \ge \int_\Omega F(\nabla u_{m+1}^\tau(x,(m+1)\tau))\,dx \ge 0,
\end{aligned}$$

and each u_k^τ, $0 \le k \le m$, at some fixed $t \in [0,T]$, satisfies

$$\frac{u_{k+1}^\tau - u_k^\tau}{\tau} = \frac{\delta F(\nabla w)}{\delta w}\,[u_{k+1}^\tau] = \nabla \cdot [\nabla u_{k+1}^\tau\, |\nabla u_{k+1}^\tau|^2] \text{ for a.e. x.}$$

□

Each u_{k+1}^{τ} can be viewed as a solution of the minimization problem (7) over a time interval of length τ. To build an interpolation $u^{\tau} : [0, T] \times \Omega \to \mathbb{R}$ of the u_k^{τ}'s, let

$$v_{k+1}^{\tau}(x,t) := \frac{u_{k+1}^{\tau}(x,(k+1)\tau) - u_k^{\tau}(x,k\tau)}{\tau}$$

and

$$u^{\tau}(x,t) = u_k^{\tau}(x,k\tau) + (t-k\tau)\, v_{k+1}^{\tau}(x,t) \text{ for } t \in]k\tau,(k+1)\tau].$$

Next set

$$v^{\tau}(x,t) = v_{k+1}^{\tau}(x,t) \text{ for } t \in]k\tau,(k+1)\tau].$$

Thus by construction,

$$\dot{u}^{\tau}(x,t) = v^{\tau}(x,t)$$

and

$$\int_{\Omega} \frac{|u_{k+1}^{\tau}(x,(k+1)\tau) - u_k^{\tau}(x,k\tau)|^2}{2\tau} dx \ + \int_{\Omega} F(\nabla u_{k+1}^{\tau}(x,(k+1)\tau))\, dx \leq \int_{\Omega} F(\nabla u_k^{\tau}(x,k\tau))\, dx. \tag{10}$$

In particular, estimates on u^{τ} and $\dot{u}^{\tau}$ in $L^2([0,T])$ can be obtained via the $\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)$ norm. Then the Arzela-Ascoli Theorem yields the following proposition.

Proposition 3.17 *Choose* $T > 0$ *and partition* $[0,T]$ *into time steps of size* $\tau > 0$. *Then for any fixed* $\tau < 1$, $u^{\tau}(x,\cdot) \in H^1([0,T])$ *for almost every* $x \in \Omega$. *In particular,* $\{u^{\tau}\}_{0<\tau\leq T}$ *converges uniformly to some* $H \in L^2([0,T])$.

Proof Note that the monotonically decreasing bounds in (8) give

$$\sum_{k=0}^{l} \int_{\Omega} \frac{|u_{k+1}^{\tau}(x,(k+1)\tau) - u_k^{\tau}(x,k\tau)|^2}{2\tau}\, dx \leq \int_{\Omega} F(\nabla u_0^{\tau}(x,t_0))\, dx - \int_{\Omega} F(\nabla u_{l+1}^{\tau}(x,(l+1)\tau))\, dx \leq ||u_0^{\tau}(x,t_0)||^4_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)} \tag{11}$$

for $0 \le l \le \lfloor \frac{T}{\tau} \rfloor$, hence

$$
\begin{aligned}
2\int_0^T |\dot{u}^\tau(x,t)|^2\,dt &= 2\sum_{k=0}^{l}\int_{k\tau}^{(k+1)\tau} |\dot{u}^\tau(x,t)|^2\,dt \\
&= 2\sum_{k=0}^{l} \tau |v_{k+1}^\tau|^2 \\
&= \sum_{k=0}^{l} \frac{|u^\tau(x,(k+1)\tau) - u^\tau(x,k\tau)|^2}{2\tau} \\
&\le ||u_0^\tau(x,t_0)||^4_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)}, \qquad (12)
\end{aligned}
$$

hence $\dot{u}^\tau \in L^2([0,T])$. To see that $u^\tau(x,\cdot)$ is in $L^2([0,T])$ for almost every fixed choice of x, observe that for $t \in [0,\tau]$, (10) implies

$$
\int_\Omega \left[F(\nabla u_1^\tau(x,k\tau)) + \frac{|u_1^\tau(x,k\tau) - u_0^\tau(x,t_0)|^2}{2\tau} \right] dx \le \int_\Omega F(\nabla u_0^\tau(x,t_0))\,dx.
$$

When $\tau < 1$,

$$
\begin{aligned}
&\int_\Omega |u_1^\tau(x,k\tau)|^2\,dx \le \int_\Omega [|u_0^\tau(x,t_0)|^2 + |u_1^\tau(x,k\tau) - u_0^\tau(x,t_0)|^2]\,dx \\
&\le \int_\Omega |u_0^\tau(x,t_0)|^2\,dx + \int_\Omega \frac{|u_1^\tau(x,k\tau) - u_0^\tau(x,t_0)|^2}{2\tau}\,dx \\
&\le \int_\Omega |u_0^\tau(x,t_0)|^2\,dx + \int_\Omega F(\nabla u_0^\tau(x,t_0))\,dx \\
&\qquad\qquad - \int_\Omega F(\nabla u_1^\tau(x,k\tau))\,dx \\
&\le \left(\int_\Omega (|u_0^\tau(x,t_0)|^2 h^{5/3})^2\,dx\right)^{1/2} \left(\int_\Omega (h^{-5/3})^2\,dx\right)^{1/2} + ||u_0^\tau(x,t_0)||^4_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)} \\
&\le ||u_0^\tau(x,t_0)||^2_{\widetilde{W}_h^{1,4}(\Omega)} (||h||_\infty^{-10/3})^{1/2} + ||u_0^\tau(x,t_0)||^4_{\overset{\bullet}{\widetilde{W}}{}_h^{1,4}(\Omega)}.
\end{aligned}
$$

Suppose that

$$
\begin{aligned}
\int_\Omega |u_k^\tau(x,k\tau)|^2\,dx \le ||u_0^\tau(x,t_0)||^2_{\widetilde{W}_h^{1,4}(\Omega)} (||h||_\infty^{-10/3})^{1/2} + \int_\Omega F(\nabla u_0^\tau(x,t_0))\,dx \\
- \int_\Omega F(\nabla u_k^\tau(x,k\tau))\,dx. \qquad (13)
\end{aligned}
$$

Then for $\tau < 1$,

$$\begin{aligned}
&\int_\Omega |u_{k+1}^\tau(x,(k+1)\tau)|^2\,dx \\
&\le \int_\Omega [|u_k^\tau(x,k\tau)|^2 + |u_{k+1}^\tau(x,(k+1)\tau) - u_k(x,k\tau)|^2]\,dx \\
&\le \int_\Omega |u_k^\tau(x,k\tau)|^2\,dx + \int_\Omega \frac{u_{k+1}^\tau(x,(k+1)\tau) - u_k(x,k\tau)|^2}{2\tau}\,dx \\
&\le \int_\Omega |u_k^\tau(x,k\tau)|^2\,dx \\
&\quad + \int_\Omega F(\nabla u_k^\tau(x,k\tau))\,dx - \int_\Omega F(\nabla u_{k+1}^\tau(x,(k+1)\tau))\,dx \\
&\le ||u_0^\tau(x,t_0)||^2_{\widetilde{W}_h^{1,4}(\Omega)}(||h||_\infty^{-10/3})^{1/2} + \int_\Omega F(\nabla u_0^\tau(x,t_0))\,dx \\
&\quad - \int_\Omega F(\nabla u_k^\tau(x,k\tau))\,dx \\
&\quad + \int_\Omega F(\nabla u_k^\tau(x,k\tau))\,dx - \int_\Omega F(\nabla u_{k+1}^\tau(x,(k+1)\tau))\,dx \\
&\le ||u_0^\tau(x,t_0)||^2_{\widetilde{W}_h^{1,4}(\Omega)}(||h||_\infty^{-10/3})^{1/2} + \int_\Omega F(\nabla u_0^\tau(x,t_0))\,dx \\
&\quad - \int_\Omega F(\nabla u_{k+1}^\tau(x,(k+1)\tau))\,dx \\
&\le ||u_0^\tau(x,t_0)||^2_{\widetilde{W}_h^{1,4}(\Omega)}(||h||_\infty^{-10/3})^{1/2} + ||u_0^\tau(x,t_0)||^4_{\overset{\bullet}{\widetilde{W}_h^{1,4}}(\Omega)},
\end{aligned}$$

hence, by induction

$$\begin{aligned}
\int_0^T |u^\tau(x,t)|^2\,dt &\le \int_0^T \int_\Omega |u^\tau(x,t)|^2\,dx\,dt \\
&\le \int_0^T 3\Big[||u_0^\tau(x,t_0)||^2_{\widetilde{W}_h^{1,4}(\Omega)}(||h||_\infty^{-10/3})^{1/2} + ||u_0^\tau(x,t_0)||^4_{\overset{\bullet}{\widetilde{W}_h^{1,4}}(\Omega)}\Big]\,dt \\
&\le 3T\left(||u_0^\tau(x,t_0)||^2_{\widetilde{W}_h^{1,4}(\Omega)}(||h||_\infty^{-10/3})^{1/2} + ||u_0^\tau(x,t_0)||^4_{\overset{\bullet}{\widetilde{W}_h^{1,4}}(\Omega)}\right). \qquad (14)
\end{aligned}$$

As a consequence, $u^\tau(x,\cdot) \in H^1([0,T])$ for almost every x.

Since $H^1([0,T])$ can be mapped to $C^{0,1/2}([0,T])$ injectively, $\{u^\tau\}_{0<\tau\le T}$ is equicontinuous over $[0,T]$. Because every $u^\tau(x,0) = u_0^\tau(x,t_0)$ and the bound in (14) holds uniformly for every τ, a subsequence can be extracted from $\{u^\tau\}_{0<\tau\le T}$ which converges uniformly to some $H \in L^2([0,T])$. Let $\{u^{\tau_{n'}}\}_{n'\in\mathbb{N}}$ denote this uniformly converging subsequence. As a consequence of the inqualities in (8), $\int_\Omega F(\nabla u^{\tau_{n'}})\,dx$ is a monotonically decreasing sequence of nonnegative real numbers. In particular, H must be a minimizer of the functional given by $u \mapsto \int_\Omega F(\nabla u)\,dx$. □

Proposition 3.18 *Let $H \in L^2([0,T])$ be as defined in Proposition 3.17. Then for all $t \in [0,T]$, $H(\cdot,t) \in \widetilde{W}_h^{1,4}(\Omega)$.*

Proof Up to further extraction of subsequences of $\{u^{\tau_{n'}}\}_{n' \in \mathbb{N}}$, the limiting curve H can be shown to be in $\widetilde{W}_h^{1,4}(\Omega)$. To see that each $u^\tau \in L^4_{h^{10/3}}(\Omega)$, recall that each $u_k^\tau \in \widetilde{W}_h^{1,4}(\Omega)$. When $t \in [k\tau, (k+1)\tau]$,

$$\begin{aligned}\int_\Omega |u^\tau(x,t)|^4 h^{10/3}\,dx &\le \int_\Omega |u_k^\tau(x,k\tau)|^4 h^{10/3} \\ &\quad + \int_\Omega |u_{k+1}^\tau(x,(k+1)\tau) - u_k^\tau(x,k\tau)|^4 h^{10/3}\,dx \\ &\le 2||u_k^\tau(x,k\tau)||^4_{\widetilde{W}_h^{1,4}(\Omega)} + ||u_{k+1}^\tau(x,(k+1)\tau)||^4_{\widetilde{W}_h^{1,4}(\Omega)}. \end{aligned} \tag{15}$$

Furthermore, (8) implies the weighted partial derivatives of each u^τ with respect to the spatial variables are also in $L^4(\Omega)$. Thus each $u^{\tau_{n'}}(\cdot,t) \in \widetilde{W}_h^{1,4}(\Omega)$ for every fixed choice of t. By the Rellich-Kondrachov Theorem, $W^{1,4}(\Omega) \subset L^4(\Omega)$ with compact injection (see Theorem 9.16 in [14]). Recall from the proof of Lemma 3.7 that $||\cdot||_{W^{1,4}(\Omega)}$ is equivalent to $||\cdot||_{\widetilde{W}_h^{1,4}(\Omega)}$ on $\widetilde{W}_h^{1,4}(\Omega)$. Consequently, $\widetilde{W}_h^{1,4}(\Omega) \subset L^4(\Omega)$ with compact injection. Since $u \mapsto \int_\Omega F(\nabla u)\,dx$ yields only nonnegative values on $\{u^{\tau_{n'}}\}_{n' \in \mathbb{N}}$, a subsequence of this family of functions can be extracted that converges to some $H \in L^4(\Omega) \cap \widetilde{W}_h^{1,4}(\Omega)$ which also minimizes this functional. □

Proposition 3.19 *The weak solution H satisfies the apriori bound*

$$||H(x,t)||^4_{\widetilde{W}_h^{1,4}(\Omega)} \le ||H_0(x,t_0)||^4_{\widetilde{W}_h^{1,4}(\Omega)}, \tag{16}$$

in $\widetilde{W}_h^{1,4}(\Omega)$.

Proof The estimates follow from eq (8) in the proof of Propositon 3.16. □

We can now prove Theorem 2.1.

Proof Propositions 3.17 and 3.18 give existence and uniqueness of $H \in (L^2([0,T]; W_h^{1,4}(\Omega))$ such that $H(x,0) = H_0(x,t_0)$. As a consequence of (9), H satisfies eq (2). To extend H uniquely to a function in $H(L^2([0,2T]; W_h^{1,4}(\Omega))$, apply the same arguments with $H(x,T)$ in the role of the initial condition $H_0(x,t_0)$ and concatenate the resulting function with H. By construction, this extension will also satisfy the apriori bound given in Proposition 3.19 and satisfy eq (2). Thus given any $k \in \mathbb{Z}$, H can be extended uniquely to $[0,kT] \times \Omega$ without escaping to infinity and this extension will also satisfy eq (2). Hence the maximum interval of existence for H is $[0,\infty)$. □

Corollary 3.20 *The weak solution is Hölder continuous, $H \in C^{1/2}[[0,\infty); C^{1/2}(\Omega)]$.*

Proof Recall from the proof of Proposition 3.17 that $H \in C^{1/2}[0, T]$. By construction, the unique extension of H to $[0, kT]$ for any $k \in \mathbb{Z}$ given in the proof of Theorem 2.1 will be in $C^{1/2}[(h-1)T, hT]$ for every $0 \le h \le k$. Thus $H \in C^{1/2}[0, \infty)$. Lastly, $H \in \widetilde{W}_h^{1,4}(\Omega)$ as a consequence of Proposition 3.19, hence $H \in C^{1-2/4}(\Omega)$. □

In the mathematical model developed by Birnir et al., the roughness of the surface is measured by the variogram, which is a structure function representing the root mean square of the elevation differences as a function of different distances of separation in a given direction [7]. Surfaces and interfaces modeled by nonlinear PDEs driven by noise (2) are often too complex for the functions arising as solutions to even possess derivatives. Therefore, the statistical structure of these surfaces is studied through their structure functions. For the computation of the variogram, see [3] and [9]. Birnir and his collaborators measure the roughness of the surface and obtain scaling results in the direction transverse to the ridge. In particular, the variogram of H is computed using one-dimensional slices (perpendicular to the downslope direction of the water surface). More precisely,

Corollary 3.21 *Let $\vec{y}$ denote the direction perpendicular to that of the upslope direction of the water surface and $\Omega_{\vec{y}}$ a one-dimensional slice of Ω in this direction. The weak solution restricted to $\Omega_{\vec{y}}$, $H|_{\Omega_{\vec{y}}}$, is Hölder continuous. In particular, $H|_{\Omega_{\vec{y}}} \in C^{1/2}[[0, \infty); C^{3/4}(\Omega_{\vec{y}})]$.*

Proof Since the gradient of the water surface is perpendicular to the upslope direction, $\Omega_{\vec{y}}$ is in the same direction. Thus the sediment flow is in this direction. In fact, water surface gradients drive both water and sediment flow and the corresponding PDEs are one-dimensional in the direction of the water surface gradient [10]. Hence the equations for the water surface when restricted to $\Omega_{\vec{y}}$ are well-posed and $H|_{\Omega_{\vec{y}}} \in C^{1-1/4}(\Omega_{\vec{y}})$. □

Numerical simulations [3] and [9] indicate that Corollary 3.21 is sharp, see also [7].

3.3 Uniqueness of weak solutions

Now consider the PDE in Birnir and Rowlett's work which describes the sediment flow (2). In Theorem 1 of [8], they demonstrate that weak solutions of (2) are unique. For $H \in \widetilde{W}_h^{1,4}(\Omega)$ and $\phi \in C_c^\infty(\Omega)$, the Cauchy-Schwartz Inequality yields

$$
\begin{aligned}
\int_\Omega \frac{\partial H}{\partial t}\phi\, dx &= \int_\Omega \nabla \cdot [\nabla H |\nabla H|^2 h^{10/3}]\,\phi\, dx \\
&= -\int_\Omega [\nabla H |\nabla H|^2 h^{10/3}] \cdot \nabla\phi\, dx \\
&\le \int_\Omega \Big| [\nabla H |\nabla H|^2 h^{5/3} h^{5/3}] \cdot \nabla\phi \Big|\, dx \\
&\le \Big(\int_\Omega \Big| |\nabla H|^2\, h^{5/3} \Big|^2 dx \Big)^{1/2} \Big(\int_\Omega |h^{5/3} \nabla H \cdot \nabla\phi|^2\, dx \Big)^{1/2}
\end{aligned}
$$

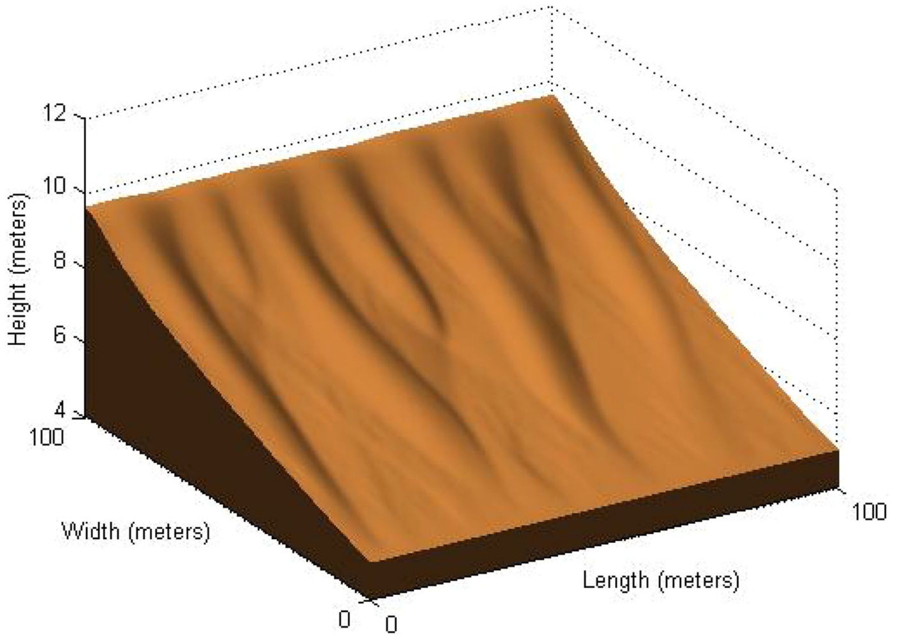

Fig. 3 A water surface with diffusion down the gradient of the free water surface H, from [9]

$$
\begin{aligned}
&\leq ||H||^2_{\widetilde{W}_h^{1,4}(\Omega)} \left(\int_\Omega |h^{5/3} \nabla H \cdot \nabla \phi|^2 \, dx \right)^{1/2} \\
&< \infty,
\end{aligned}
$$

thereby verifying that H is a weak solution of (2). For any $\phi \in C_0^\infty[0, \infty) \times C_0^\infty(\Omega)$, compactly supported, integration by parts yields

$$
\int_0^\infty \int_\Omega H \frac{\partial \phi}{\partial t} \phi \, dx \, dt = \int_0^\infty \int_\Omega [\nabla H |\nabla H|^2 h^{10/3}] \cdot \nabla \phi \, dx \, dt,
$$

and it is straightforward to show using the above estimates that both sides are finite. Thus, H is the unique function in $\widetilde{W}_h^{1,4}(\Omega)$ which minimizes $\int_\Omega F(\nabla u) \, h dx$.

4 Diffusion of the water and the land surfaces

In landscape evolution, with the right environmental conditions, gentle slope, saturated soil, the water and sediment can diffuse down the water surface H or the soil can diffuse down the land surface z. The latter diffusion is also called creep. These two types of diffusions are different and they give rise to different looking surfaces, compare Fig 3 and 4. The model eq (2) can be adapted to include the diffusion by adding in a diffusion term for the water and land surfaces. For the diffusing water surface the equation becomes

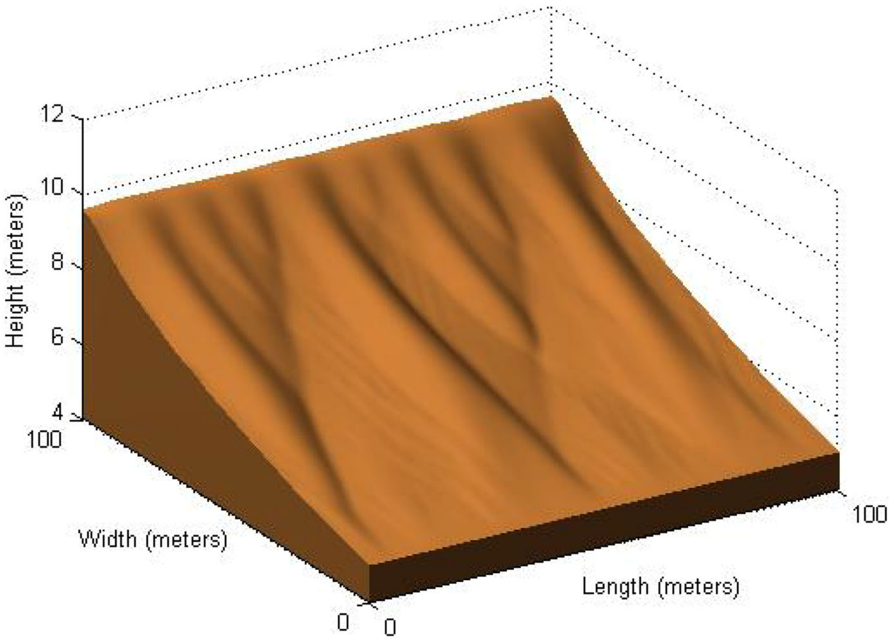

Fig. 4 A water surface with diffusion down the gradient of the land surface z, from [9]

$$\frac{\partial H}{\partial t} = \nabla \cdot [\nabla H |\nabla H|^2 h^{10/3}] + \beta \Delta H, \tag{17}$$

where β is the diffusion coefficient, see Fig 3. The height of the landsurface is described by

$$z(x, y, t) := H(x, y, t) - h(x, y).$$

Thus the sediment flow with diffusing land surface is modeled by

$$\frac{\partial H}{\partial t} = \nabla \cdot [\nabla H |\nabla H|^2 h^{10/3}] + \beta \Delta z, \tag{18}$$

where β again denotes the diffusion coefficient. The presence of diffusion, in either case, leads to the existence of strong solutions to respectively the eq (17) and (18), for $t > 0$. The construction of such solutions is achieved by convolution with the heat kernel. To encode the boundary conditions (1), take the Green's function given by

$$G(x, y, x', y', t) = 2 \sum_{\substack{n=1 \\ 2m=-\infty}}^{\infty} \sin\left(\frac{\pi n x}{W}\right) \sin\left(\frac{\pi n x'}{W}\right) e^{\frac{2\pi i m}{L}(y-y')} e^{-2\beta\pi^2\left(\frac{n^2}{W^2}+\frac{4m^2}{L^2}\right)t}.$$

To study regularity of solutions in the presence of diffusion terms, we begin by considering the following integral representation for the water surface.

Lemma 4.1 *Let $h(x, y)$ be a given function which satisfies the conditions described in (3). Given $H_0 \in W_h^{1,4}(\Omega)$, suppose there exists $H \in L^2([0,\infty); W_h^{1,4}(\Omega))$ which satisfies the boundary conditions and such that*

$$H = H_0 \text{ a. e. on } \Omega, \qquad t = 0;$$
$$\int_\Omega \Big(\frac{\partial H}{\partial t}\phi + h^{10/3}|\nabla H(t)|^2 \nabla H(t)\cdot\nabla\phi\Big)\,dx \,-\, \beta\int_\Omega \Delta H(t)\phi\,dx = 0, \qquad t \in (0,T);$$

for all $\phi \in C_0^\infty(\Omega \setminus h^{-1}(0))$. In particular, assume that H is a weak solution of

$$\frac{\partial H}{\partial t} = \nabla\cdot[\nabla H|\nabla H|^2 h^{10/3}] + \beta\Delta H.$$

Then

$$H(x,t) = G(t) * H_0 + \int_0^t G(s) * \nabla\cdot(|\nabla H|^2 \nabla H\, h^{10/3})\,ds,$$

is smooth in x and t (in $C^\infty((0,\infty); C^\infty(\Omega))$), for $t > 0$.

Proof To describe mixed spatial derivatives, let $\alpha \in \mathbb{N}_0^n$ denote a multi-index, $\alpha = (k, j)$, $\partial_x^\alpha = \partial_{x_1}^k \partial_{x_1}^k$, $k + j = n$ and $x_1 = x$, $x_2 = y$. For every $n \in \mathbb{Z}^+$, the Divergence Theorem implies

$$\partial_x^\alpha H = \partial_x^\alpha G * H_0 - \int_0^t \partial_x^\alpha \nabla G(s) * (|\nabla H|^2 \nabla H\, h^{10/3})\,ds \quad \in \quad C^0((0,\infty); C^0(\Omega)),$$

hence

$$\partial_t\partial_x^\alpha H = \partial_t\partial_x^\alpha G * H_0 - \partial_x^\alpha \nabla G(s) * (|\nabla H|^2 \nabla H\, h^{10/3})\,(x,t) \quad \in \quad C^0((0,\infty); C^0(\Omega)),$$

for $t > 0$. A straightforward induction argument now gives that for $t > 0$,

$$\partial_t^k\partial_x^\alpha H = \partial_t^k\partial_x^\alpha G * H_0 - \partial_t^{k-1}\partial_x^\alpha \nabla G(s) * (|\nabla H|^2 \nabla H\, h^{10/3})\,ds \quad \in \quad C^0((0,\infty); C^0(\Omega)).$$

□

4.1 Existence and uniqueness of weak solutions in the presence of diffusion terms

An additional condition is needed on the water level h to obtain existence of weak solutions when diffusion terms are added to model eq (2). More precisely,

Proposition 4.2 *Let $h(x, y)$ be a given function which satisfies the conditions described in (3). Then there exists a unique weak solution of model eq (17). In addition, suppose that $\nabla h \in L^2(\Omega)$. Then, for any $H_0 \in W_h^{1,4}(\Omega)$, there exists a unique $H \in L^2([0,\infty); W_h^{1,4}(\Omega))$ which satisfies the boundary conditions and such that*

$$H = H_0 \text{ a. e. on } \Omega, \qquad t = 0;$$
$$\int_\Omega \Big(\frac{\partial H}{\partial t}\phi + h^{10/3}|\nabla H(t)|^2 \nabla H(t)\cdot\nabla\phi\Big)\,dx \;-\; \beta\int_\Omega \Delta(H(t)-h)\phi\,dx = 0, t\in(0,T);$$

for all $\phi \in C_0^\infty(\Omega \setminus h^{-1}(0))$. In particular, there exists a unique weak solution of model eq (18) and this solution is given by H.

Proof The first statement follows by the following modification of the functional K in Sect 3.2

$$K^\tau = K + \int |\nabla u|^2\,dx.$$

This functional is sequentially weakly lower semicontinuous by the arguments in Sect 3.2.

The same suite of arguments used in Sect 3.2 to prove Theorem 2.1 can be applied if

$$K^\tau \ + \beta \int N(x, u, \nabla(u-h))\,dx$$

can be shown to be sequentially weakly lower semicontinuous. Observe that $(p, q) \to F^\tau(x, p, q) + \alpha|q - \nabla h|^2$ can be viewed as convex for all $p \in \mathbb{R}$ and almost every $x \in \Omega$. Recall that bounds on F_q^τ in $L^{4/3}(\Omega)$ and F_p^τ in $L^2(\Omega)$ were achieved in the proof of Lemma 3.14. Since

$$0 \le N(x, p, q - \nabla h) \le 2|q|^2 + 2|\nabla h|^2,$$

it suffices to to obtain a bound on $N_q(x, v, \nabla v - \nabla h)$ in $L^2(\Omega)$. Because of the additional assumption that $\nabla h \in L^2(\Omega)$, there exist $C_5, C_6 > 0$ such that

$$
\begin{aligned}
\int_\Omega |N_q(x, v, \nabla(v-h))|^2\,dx &\le C_5 \int_\Omega (1 + |\nabla v|^2 + |\nabla h|^2)\,dx \\
&\le C_6\Big(\int_\Omega (1+|\nabla h|^2)\,dx \\
&+ \Big(\int_\Omega |\nabla v|^4 h^{10/3}\,dx\Big)^{1/2}\Big)\Big(\int_\Omega h^{-10/3}\,dx\Big)^{1/2} \\
&\le C_6\Big(\int_\Omega (1+|\nabla h|^2)\,dx + ||v||^2_{\widetilde{W}_h^{1,4}(\Omega)} (||h||_\infty^{-10/3})^{1/2}\Big) \\
&< \infty,
\end{aligned}
$$

as desired. Since repeated application of the Cauchy-Schwartz Inequality yields

$$
\begin{aligned}
\int_\Omega \Delta(H(t)-h)\phi\,dx &= -\int_\Omega \nabla(H(t)-h)\cdot\nabla\phi\,dx \\
&\le \int_\Omega \Big|\nabla H\cdot\nabla\phi\Big|\,dx + \int_\Omega \Big|\nabla h\cdot\nabla\phi\Big|\,dx \\
&\le \Big(\int_\Omega \Big|\nabla H|^4 h^{10/3}\,dx\Big)^{1/4} \\
&\quad \Big(\int_\Omega \Big|\nabla H|^4 h^{10/3}\,dx\Big)^{1/4}\Big(\int_\Omega h^{-10/3}\,dx\Big)^{1/4}\Big(\int_\Omega \Big|\nabla\phi\Big|^4\,dx\Big)^{1/4} \\
&\quad + \Big(\int_\Omega \Big|\nabla h\Big|^2\,dx\Big)^{1/2}\Big(\int_\Omega \Big|\nabla\phi\Big|^2\,dx\Big)^{1/2} \\
&\le ||H||^2_{\widetilde{W}_h^{1,4}(\Omega)} (||h||_\infty^{-10/3})^{1/4}\Big(\int_\Omega \Big|\nabla\phi\Big|^4\,dx\Big)^{1/4} \\
&\quad + ||\nabla h||_{L^2(\Omega)}\Big(\int_\Omega \Big|\nabla\phi\Big|^2\,dx\Big)^{1/2} \\
&< \infty
\end{aligned}
$$

for $H \in W_h^{1,4}(\Omega)$, $\nabla h \in L^2(\Omega)$, and $\phi \in C_c^\infty(\Omega)$, arguments used for the case of model eq (2) in Sect 3.2 can be extended to show that H weakly solves model eq (18). Uniqueness of weak solutions for model eq (2) follows from a slight modification of the proof of Theorem 3 in [8], see below. □

Theorem 4.3 *(Birnir and Rowlett [8]) Assume that H and F are two weak solutions of (17) or (18). Then H and F are equal as elements of $C^1([0,\infty); L^2(\Omega))$.*

Proof We differentiate the L^2 norm of $H - F$ with respect to time and get, for either eq (17) or (18), that

$$
\frac{d}{dt}\|H-F\|_2^2 = \int_\Omega \Big(\nabla\cdot[(\nabla H|\nabla H|^2 - \nabla F|\nabla F|^2)h^{10/3} + \beta\nabla(H-F)](H-F)\Big)\,dx
$$

because the term $-\beta\Delta h$ washes out in the eq (18). An application of the Divergence Theorem gives that

$$\begin{aligned}
&\frac{d}{dt}\|H-F\|_2^2\\
&=-2\int_\Omega\left([|\nabla H|^4+|\nabla F|^4-\langle\nabla H,\nabla F\rangle(|\nabla H|^2+|\nabla F|^2)]h^{10/3}+\frac{\beta}{2}|\nabla(H-F)|^2\right)dx\\
&\le -2\int_\Omega\left([|\nabla H|^4+|\nabla F|^4-|\nabla H||\nabla F|(|\nabla H|^2+|\nabla F|^2)]h^{10/3}+\frac{\beta}{2}|\nabla(H-F)|^2\right)dx,
\end{aligned}$$

by the point-wise Schwarz inequality applied to $\langle\nabla H,\nabla F\rangle$. The application of the Divergence Theorem is easily justified by approximating H and F by smooth functions. Now it is easy to show that, for all real numbers a and b,

$$a^4+b^4-ab(a^2+b^2)\ge 0,$$

so

$$\frac{d}{dt}\|H-F\|_2^2\le 0,$$

since $\beta>0$. Now H and F have the same initial data, so

$$\|H-F\|_2^2(t)=0,\quad \text{for all } t>0.$$

□

4.2 Regularity results for the land surface

Building on the regularity results for the water surface, we now show that the land surface is smooth with diffusion in z,

Lemma 4.4 *Let $h(x,y)$ be a given function which satisfies the conditions described in (3). In addition, suppose that $\nabla h\in L^2(\Omega)$. Given $H_0\in W_h^{1,4}(\Omega)$, let $H\in L^2([0,\infty);W_h^{1,4}(\Omega))$ be the unique weak solution of model eq (18) which satisfies the boundary conditions and such that*

$$H=H_0 \text{ a. e. on } \Omega,\qquad t=0.$$

Then the land surface z is smooth in x and t (in $C^\infty((0,\infty);C^\infty(\Omega))$), for $t>0$.

Proof If H is a weak solution of model eq (18), then the integral representation of H is given by

$$H=G(t)*H_0-\int_0^t\nabla G(s)*(|\nabla H|^2\nabla Hh^{10/3})\,ds-\beta\int_0^t G*\Delta h\,ds$$

Since G satisfies the heat equation,

$$\beta \int_0^t G * \Delta h \, ds = \beta \int_0^t \Delta G * h \, ds = \int_0^t \partial_t G * h \, ds.$$

Furthermore,

$$\int_0^t \partial_t G * h \, dt = \int_0^t \partial_t G \, dt * h = G(t) * h \; - \; G(0) * h = G(t) * h \; - \; h,$$

where the first equality follows because h does not depend on t and last equality from the fact that $G(0) = \delta(0)$, where δ denotes the Dirac distribution. Thus

$$\begin{aligned} H \; &= G(t) * H_0 - \int_0^t \nabla G(t) * (|\nabla H|^2 \nabla H h^{10/3}) \, ds - G(t) * h \; + \; h, \\ &= G(t) * (H_0 - h) - \int_0^t \nabla G(t) * (|\nabla H|^2 \nabla H h^{10/3}) \, ds + \; h, \end{aligned}$$

or

$$z = H - h = G(t) * z_0 - \int_0^t \nabla G(t) * (|\nabla H|^2 \nabla H h^{10/3}) \, ds.$$

By the same arguments as in the proof of Lemma 4.1, $\partial_t^k \partial_x^\alpha z \in C^0((0, \infty); C^0(\Omega))$ for every $k \in \mathbb{N}_0$ and mult-index $\alpha \in \mathbb{N}_0^n$, $n \in \mathbb{Z}^+$, for $t > 0$. □

5 Conclusion

Mathematical models of erosion should be judged by how well they capture observable phenomena of eroding surfaces and their drainage basins. In particular, one wants these models to capture: 1. the emergence of channels on smooth surfaces and the formation of drainage pattern on these surfaces, 2. the development of long-lived surfaces forming a pattern of mountain ridges and valleys, 3. the gradual decline of the mountains and dissipation of the above forms, 4. the scaling of the long-lived landforms giving rise to Hack's law [3], and 5. the variation of these landforms depending on different environmental conditions. Three types of different models have historically been used to capture these properties, beginning with the work of Horton [5]. These are: (i) Deterministic models based on conservation principles. (ii) Stochastic models based on conservation principles. (iii) Deterministic models based on variational principles and obtained by minimizing or maximizing an aggregate quantity. The evidence based on the work so far and published in the papers [3, 7–9], is that these three approaches are in fact all related and together give a more complete picture of the erosion process.

In this paper, we showed that a deterministic, transport-limited model (2), see [1, 9], that satisfies the above requirements, has unique weak solutions when solved with the background of an equilibrium water depth h. Since h has shocks frozen in the flow, h

can only be assumed to be a measureable function, that is neither VMO nor of bounded mean oscillations (BMO). This implies that the water surface H is not smooth. We get Hölder continuous functions, with a lower bound on the scaling. This lower bound is sharp by the numerical evidence [9] for Hack's law, which gives an upper bound. Once this result is obtained, it is easy to extend it to models with diffusion either in the water surface H or in the land surface z. Interestingly, any amount of diffusion makes these surfaces smooth, see Sect 4.

The numerical method used to simulate the land surfaces, see [9], is still too slow. But hopefully, faster methods can be found using optimal transport. This will be pursued in a future publication.

Author Contributions Both authors wrote the main manuscript text and also reviewed the manuscript.

Data Availability No datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

Ethical approval Not applicable.

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